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LETTER TO THE EDITOR

A geometric phase for $m = 0$ spins

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Abstract. A $|jm\rangle$ spin state in an adiabatically-cycled magnetic field acquires a geometric phase of m times the solid angle described by $\boldsymbol{B}$, so that for $m = 0$ states the geometric phase vanishes. However, if $\boldsymbol{B}$ is not cycled, but is made to reverse direction, an $m = 0$ state returns to itself and in so doing acquires a geometric phase factor of $(-1)^j$. This phase is of a topological character; parameter space is the real projective plane, in which the phase distinguishes trivial from non-trivial cycles.

A spin-j particle in a slowly changing magnetic field is a canonical example of the geometric phase (Berry 1984). The spin Hamiltonian $\hat{H}(\boldsymbol{B})$ is equal to $\boldsymbol{B}\cdot\hat{\boldsymbol{J}}$, where $\hat{\boldsymbol{J}}$ is the spin angular momentum. The energy levels $E_m(\boldsymbol{B}) = mB\hbar$ and eigenstates $|jm(\boldsymbol{B})\rangle$ are determined by $\boldsymbol{B}$. If initially the spin is in the mth eigenstate and $\boldsymbol{B}$ is slowly taken round a closed cycle C, then (neglecting transitions) the spin remains in the mth state of the changing Hamiltonian and returns to itself up to a phase factor, part of which includes the geometric phase

$$\gamma_m(C) = -\int_S \boldsymbol{V}_m\cdot \mathrm{d}\boldsymbol{S} \qquad \text{where } \boldsymbol{V}_m = m\frac{\boldsymbol{B}}{B^3}. \tag{1}$$

The integral is taken over a surface S (in $\boldsymbol{B}$-space) whose boundary is C. $\gamma_m(C)$ is easily seen to be m times the solid angle subtended by C with respect to $\boldsymbol{B} = 0$.

From (1), it follows that the geometric phase vanishes for $m = 0$. The purpose of this letter is to point out the existence of a geometric phase for $m = 0$ states when the magnetic field is taken not through a cycle but rather a half-cycle, in which $\boldsymbol{B} \longrightarrow -\boldsymbol{B}$. Since $|j\,m(-\boldsymbol{B})\rangle = (\text{phase factor}) \times |j\ -m(\boldsymbol{B})\rangle$, it follows that

$$|j\,0(-\boldsymbol{B})\rangle = (\text{phase factor}) \times |j\,0(\boldsymbol{B})\rangle. \tag{2}$$

Therefore, if $\boldsymbol{B}$ is slowly turned to $-\boldsymbol{B}$, the state $|j\,0(\boldsymbol{B})\rangle$ returns to itself up to a purely geometrical phase factor (the dynamical phase $\int E_m(\boldsymbol{B}_t)/\hbar\,\mathrm{d}t$ vanishes for $m = 0$.)

Because $\boldsymbol{B}(t)$ does not close for a half-cycle, we cannot compute $\gamma_{m=0}$ from (1), in which the phase is expressed as a flux through a closed circuit in $\boldsymbol{B}$-space. Instead we obtain γ_0 as the phase accumulated by a parallel-transported representation of the evolving state. For the half-cycle H in which $\boldsymbol{B}$ is rotated about the y-axis from $B\hat{\boldsymbol{z}}$ to $-B\hat{\boldsymbol{z}}$ at constant angular velocity π/T, such a representation is given by

$$|\psi(t)\rangle = \exp\left(-\mathrm{i}\frac{\hat{J}_y}{\hbar}\frac{\pi t}{T}\right)|j\,0\rangle \tag{3}$$

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where $|j\,0\rangle$ is the $m = 0$ eigenstate of $\hat{J}_z$, because

$$\langle\psi(t)|\dot{\psi}(t)\rangle = \text{const.} \times \langle j\,0|\hat{J}_y|j\,0\rangle = 0. \tag{4}$$

Therefore the accumulated phase is given by

$$\begin{aligned} e^{i\gamma_0(H)} &= \langle\psi(0)|\psi(T)\rangle \\ &= \langle j\,0|\exp(-i\pi\hat{J}_y/\hbar)|j\,0\rangle = (-1)^j. \end{aligned} \tag{5}$$

The last equality in (5) is obtained by noting that (i) because the spherical harmonic $Y_{j0}(\theta,\phi)$ (which corresponds to $|j\,0\rangle$) is rotationally symmetric about $\hat{z}$, a π-rotation about $\hat{y}$ (under which $x \to -x$ and $z \to -z$) has the same effect as the reflection $\boldsymbol{r} \to -\boldsymbol{r}$, and (ii) $Y_{j0}(\theta,\phi)$ has parity $(-1)^j$. (Alternatively, one can observe that the second-to-last member of (5) is the Wigner D-matrix element $D^j_{00}(0,\pi,0)$, which is equal to $(-1)^j$ according to a standard formula (see e.g. Sakurai 1985).)

Equation (5) is our principal result. Note that for an $m = 0$ state to exist, j must be integral; for the phase to be non-trivial, j must be odd.

Although (5) was obtained for the particular half-cycle H, it is easy to see that the geometric phase factor $(-1)^j$ is the same for an arbitrary half-cycle H'. To see this, note that by applying a fixed rotation and rescaling the magnitude of $\boldsymbol{B}$ along H' (neither of which changes the geometric phase), we may assume that H', like H, takes $\boldsymbol{B}$ from $B\hat{z}$ to $-B\hat{z}$. Now consider the cycle C obtained by taking $\boldsymbol{B}$ from $B\hat{z}$ to $-B\hat{z}$ along H' and then from $-B\hat{z}$ to $B\hat{z}$ along $-H$ (i.e. the reverse of H), as in figure 1. Clearly $\gamma_0(C) = \gamma_0(H') - \gamma_0(H)$. Also, since C is closed, $\gamma_0(C) = 0$, from (1). Therefore $\gamma_0(H') = \gamma_0(H)$.

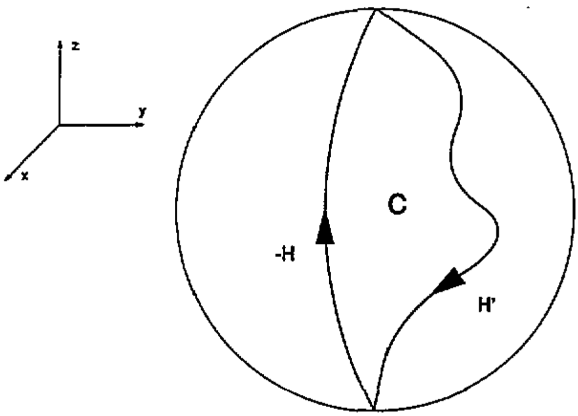

Figure 1. The cycle C on the sphere $||\boldsymbol{B}|| = B$ is composed of the two half-cycles H' and $-H$.

It follows that γ_0 is essentially topological, i.e. insensitive to smooth deformations of the path in $\boldsymbol{B}$-space. The underlying topology is elucidated by considerations of the parameter space (each point of which determines the $m = 0$ eigenstate up to a phase.) While for $m \neq 0$ states, the parameter space M is $\{\boldsymbol{B}\} - \boldsymbol{0}$ (the origin is excluded because the eigenstates are degenerate there), for $m = 0$ states we can, according to (2), take parameter space to be $\overline{M} = (\{B\} - 0)/(B \sim -B)$, i.e. three-space with the origin excluded and points related by reflection identified. Alternatively, we can fix the magnitude of B (the eigenstates depend only on its direction), and take parameter space to be $P = S^2/(B \sim -B)$, i.e. the sphere $||B|| = B$ with antipodal points identified, or equivalently, the real projective plane.

Both $\overline{M}$ and P are non-orientable, and both are doubly connected. That is, there are just two topologically distinct classes of cycles, namely trivial (i.e. contractible) cycles taking B to B and non-trivial ones taking B to $-B$. The geometric phase factor is $(-1)^j$ for non-trivial cycles and 1 for trivial ones.

A different topology underlies the well known sign change in the eigenstates of real (i.e. time-reversal invariant) Hamiltonians transported round a degeneracy (Herzberg and Longuet-Higgins 1963). An example is given by the $j = 1/2$ spin Hamiltonian, provided we fix $B_y = 0$. Then parameter space is the punctured (B_x, B_z) plane (the point of degeneracy at the origin is removed), and, from (1), the geometric phase factor is $(-1)^m$ or $(+1)^m$, according to whether a cycle encloses the origin an odd or even number of times. The topology of the punctured plane is different from that of the projective plane. While the punctured plane is multiply connected (cycles can be classified by winding number), the projective plane, as explained above, is doubly connected. A consequence of this difference is that the degeneracy phase can be expressed as the line integral round a cycle of a (continuous) vector potential A_m (in the present case $A_m = m(B_z, 0, -B_x)/B^2$), whereas the $m = 0$ phase cannot.

The space $\overline{M}$ appears in a quite different context, namely in the treatment of identical particles in quantum mechanics (Laidlaw and DeWitt 1971, Leinaas and Myrheim 1977). Consider two particles in three dimensions with coordinates r_1 and r_2, and introduce the centre-of-mass and relative coordinates, $R = (r_1 + r_2)/2$ and $r = r_1 - r_2$, respectively. If the particles are identical, the configuration (r_1, r_2) is the same as (r_2, r_1), so that r is to be identified with $-r$. If we exclude the point $r = 0$ (the particles are not allowed to coincide), we find that the relative coordinate space is just $\overline{M}$ (with B replaced by r).

In their seminal analysis of identical particles in quantum mechanics, Leinaas and Myrheim (1977) showed that the spin-statistics theorem (in three dimensions) could be derived from the following postulate: when transported around a non-trivial cycle in the relative coordinate space $\overline{M}$, a two-particle spinor, in an eigenstate of total spin angular momentum $\hat{J}^2 = (\hat{S}_1 + \hat{S}_2)^2$ with eigenvalue $j(j+1)$, should acquire a sign factor $(-1)^j$. That Leinaas and Myrheim's sign change is the same as the geometric phase calculated in (5) (for $m = 0$) suggests the possibility that it can be derived rather than postulated. Such a derivation has not yet been found.

In this discussion we have emphasized adiabatic cycles. As shown by Aharonov and Anandan (1987), the adiabatic assumption is not necessary, and the geometric phase (5) arises in any (non-trivial) cyclic evolution from $|j\,0(B)\rangle$ to $|j\,0(-B)\rangle$.

The $m = 0$ geometric phase could be observed. One way is with a beam of spin-1 atoms, polarized in the $m = 0$ state along a field B. The beam is split and inserted into an interferometer. In one arm, B is uniform. Along the other, B is twisted so that its directions at the end and beginning differ by θ. As θ is increased from 0 to π, the fringes should first disappear (when $\theta = \pi/2$) and then reappear shifted by a half-spacing. It might seem peculiar that an $m = 0$ state, whose interaction energy with B is always zero, could nevertheless be rotated by B—grasping a ghost, as it were. But there is no paradox, because for these $m = 0$ states the direction of B is a unique axis of circular symmetry and (as detailed analysis confirms) this axis does turn adiabatically with B.

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